AM205 HW0. Introduction

P1. Chebyshev polynomials

The Chebyshev polynomials $T_k(x)$ can be defined using the recursive relation

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

and $T_0(x) = 1$, $T_1(x) = x$. Evaluate and plot the Chebyshev polynomial of degree 5 at 101 evenly spaced points in the interval $x \in [-1, 1]$. Draw a 3D surface plot of the function $T_3(x)T_5(y)$ on a 101×101 grid in the domain $(x, y) \in [-1, 1]^2$.

P2. Square root

Use the iteration

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$$

to approximate $\sqrt{a}$. This is known as Heron's formula and it is equivalent to the Newton–Raphson method for the function $f(x) = x^2 - a$. Choose an initial starting value of $x_0 = a$ and iterate until $|x_{k+1} - x_k| < \epsilon$ for some tolerance ϵ. Determine the number of iterations required to compute $\sqrt{5}$ for the cases of $\epsilon = 10^{-3}$ and $\epsilon = 10^{-9}$.

P3. Finite differences

(a) Let $f(x) = \tan x$ and consider the finite-difference approximation

$$f'_a(x; h) = \frac{f(x+h) - f(x-h)}{2h}.$$

Make a log–log plot of the relative error $E = |f'(x) - f'_a(x; h)|/|f'(x)|$ at $x = 1$ as a function of h for $h = 10^{-k}$, using $k = 1, 1.5, \ldots, 16$. Use linear regression to fit the relative error to the straight line

$$\log E = \log C + q \log h$$

for some constants C and q. Show that $q \approx 2$, meaning that the approximation is second-order accurate.

(b) Repeat the analysis for the approximation

$$f'_b(x; h) = \frac{-11f(x) + 18f(x+h) - 9f(x+2h) + 2f(x+3h)}{6h}$$

and determine the rate of convergence q.

P4. Calculation of Pi

In the first lecture we discussed Archimedes' method of calculating π from perimeters of inscribed and superscribed regular polygons. Areas of the polygons can serve the same purpose. Consider a circle of unit radius. Let a_n and b_n be the areas of inscribed and superscribed regular polygons with 3×2^n sides, respectively.

(a) The case of $n = 0$ therefore corresponds to inscribed and superscribed equilateral triangles. Use geometry to show that $a_0 = \frac{3}{4}\sqrt{3}$ and $b_0 = 3\sqrt{3}$.

(b) Show that

$$\frac{2}{b_{n+1}} = \frac{1}{a_{n+1}} + \frac{1}{b_n}, \qquad a_{n+1}^2 = a_n b_n$$

and write a program to evaluate (a_n, b_n) for $n = 0, 1, \ldots, 40$. In addition, calculate $c_n = \frac{1}{2}(a_n + b_n)$.

(c) Make a log-linear plot of the absolute errors $|a_n - \pi|$ and $|c_n - \pi|$ as a function of n. How fast do these two sequences converge to π? Is there a difference in the convergence rate between a_n and c_n?