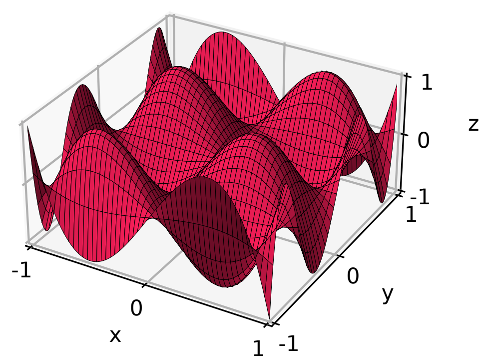
AM205 HW0. Introduction. Solution

The source code for this solution is available at
https://code.harvard.edu/AM205/public/tree/main/homework/hw0_intro/solution

P1. P1. Chebyshev polynomials

The program [p1_chebyshev.py] produces the following plot using the `mplot3d` toolkit from `Matplotlib`. The style is customized by definitions in file `matplotlibrc` automatically read from the working directory.



P2. P2. Square root

See solution code in [p2_sqrt.py].
Exact value with double precision: $\sqrt{5} \approx 2.23606797749978981$.
Computed approximations (correct digits are highlighted):

k	x_k	$\lvert x_{k+1} - x_k \rvert$
0	5	2
1	3	0.66666666666666519
2	2.33333333333333348	0.09523809523809523438
3	2.23809523809523814	0.00202634245187471862
4	2.23606889564336342	9.18143385320036032e-07
5	2.23606797749997810	1.88293824976426549e-13
6	2.23606797749978981	0

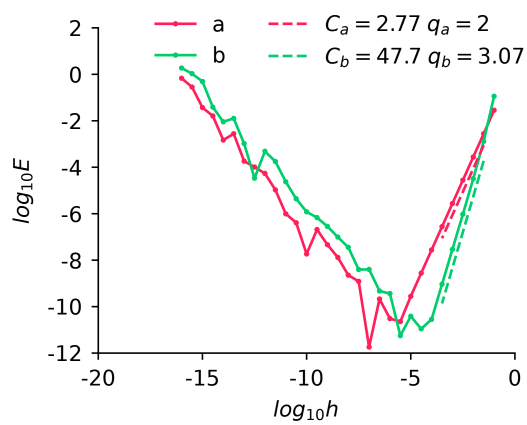
The number of iterations is 4 for $\epsilon = 10^{-3}$ and 5 for $\epsilon = 10^{-9}$.

P3. P3. Finite differences

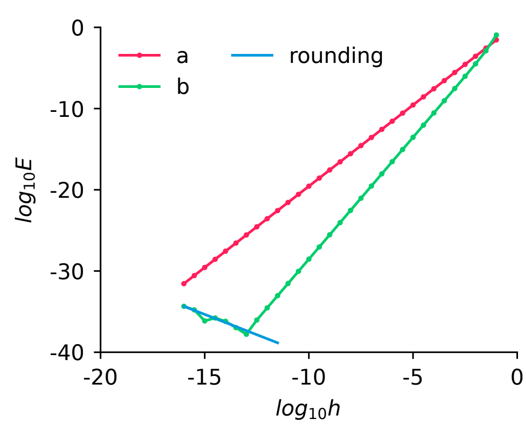
(a), (b) The approximations are implemented in [p3_fd.py]. The log-log plot of the relative error is V-shaped since the rounding error dominates for values of $h < 10^{-8}$. To analyze the accuracy of the finite difference approximations, the fitting range for the linear regression needs to be chosen in the region $h > 10^{-8}$, where the truncation error dominates. Linear regression in the range $h \in [3.16 \cdot 10^{-4}, 3.16 \cdot 10^{-2}]$ results in

$$\text{(a) } C_a = 2.77, \quad q_a = 2$$
$$\text{(b) } C_b = 47.7, \quad q_b = 3.07$$

so the approximation (a) is second-order accurate and (b) is third-order accurate. Note that values of C_a and C_b may be sensitive to the fitting range.



In addition, [p3_fd_mpmath.py] implements the same approximations using the arbitrary precision library mpmath. The following plot is produced with a precision of 50 decimal places.



P4. P4. Calculation of Pi

(a) Consider a coordinate system with the origin at the center of the circle. The inscribed triangle has vertices

$$x_0 = (1, 0), \quad x_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad x_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

Its area can be calculated using the three-dimensional vector product,

$$a_0 = \frac{|(x_1 - x_0) \times (x_2 - x_0)|}{2} = \frac{|(-\frac{3}{2}, \frac{\sqrt{3}}{2}, 0) \times (-\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0)|}{2} = \frac{|(0, 0, \frac{3\sqrt{3}}{2})|}{2} = \frac{3\sqrt{3}}{4}.$$

By similarity, the vertices of the superscribed triangle are obtained by scaling from the vertices of the inscribed triangle. The correct scaling factor is 2, giving vertices

$$\hat{x}_0 = (2, 0), \hat{x}_1 = (-1, \sqrt{3}), \hat{x}_2 = (-1, -\sqrt{3}),$$

since the edge between $\hat{x}_1$ and $\hat{x}_2$ must exactly touch the circle at $(\hat{x}_1 + \hat{x}_2)/2 = (-1, 0)$. Therefore, the area is scaled by 4 giving $b_0 = 4a_0 = 3\sqrt{3}$.

(b) Consider a regular inscribed polygon with $k = 3 \times 2^n$ sides. It can be broken down into $2k$ right triangles each with area $\frac{1}{2} \cos \frac{\pi}{k} \sin \frac{\pi}{k}$ and hence

$$a_n = k \cos \frac{\pi}{k} \sin \frac{\pi}{k}.$$

The corresponding superscribed polygon can be broken down into $2k$ right triangles each with area $\frac{1}{2} \tan \frac{\pi}{k}$ and hence

$$b_n = k \tan \frac{\pi}{k}.$$

These results agree with **(a)**

$$a_0 = 3 \cos \frac{\pi}{3} \sin \frac{\pi}{3} = \frac{3\sqrt{3}}{4} \quad \text{and} \quad b_0 = 3 \tan \frac{\pi}{3} = 3\sqrt{3}.$$

Then

$$\frac{2}{b_{n+1}} = \frac{1}{k \tan \frac{\pi}{2k}} = \frac{1 + \cos \frac{\pi}{k}}{k \sin \frac{\pi}{k}} = \frac{1}{k \sin \frac{\pi}{k}} + \frac{\cos \frac{\pi}{k}}{k \sin \frac{\pi}{k}} = \frac{1}{2k \sin \frac{\pi}{2k} \cos \frac{\pi}{2k}} + \frac{1}{b_n} = \frac{1}{a_{n+1}} + \frac{1}{b_n}$$

using the half-angle $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$ and double angle $\sin 2\theta = 2 \sin \theta \cos \theta$ identities. Similarly,

$$a_{n+1}^2 = 4k^2 \cos^2 \frac{\pi}{2k} \sin^2 \frac{\pi}{2k} = k^2 \sin^2 \frac{\pi}{k} = k^2 \tan \frac{\pi}{k} \sin \frac{\pi}{k} \cos \frac{\pi}{k} = a_n b_n$$

(c) The program [p4_pi.py] calculates the values of a_n and c_n up to $n = 40$ and estimates that

$$|a_n - \pi| \approx 2.29 \cdot 4^{-n}.$$

Both a_n and c_n have the same convergence rate.