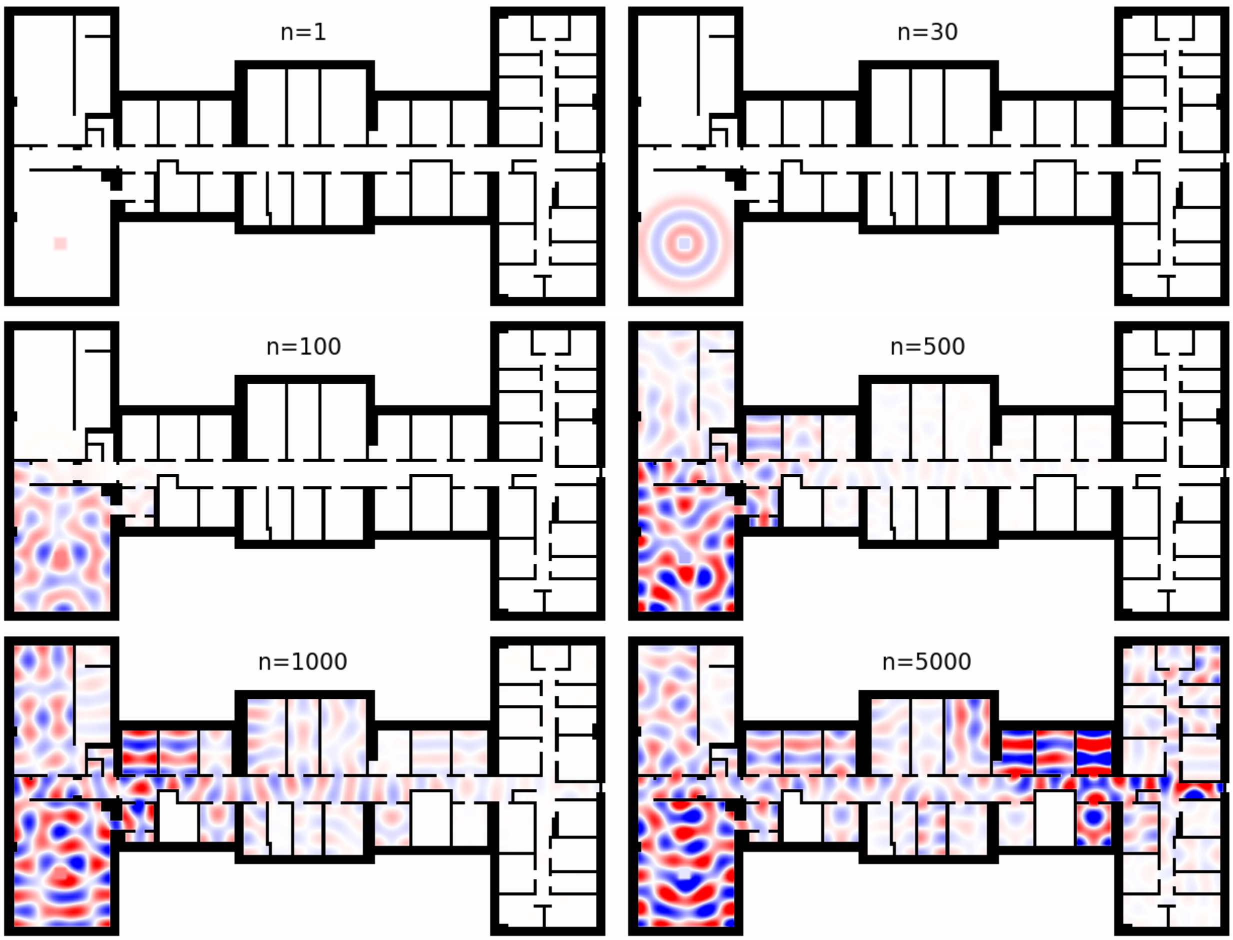
AM205 HW5. PDEs, optimization, iterative methods. Solution

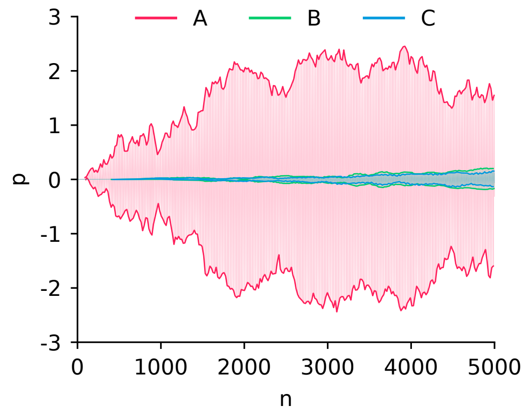
P1. Acoustics of Pierce Hall

See solution code in [p1_pierce.py].

(a,b,c) The images below show the pressure field at the requested time steps. The color map corresponds to values between $p = -2$ (blue), 0 (white), and 2 (red). The pressure waves propagate from the speaker. They remain radially symmetric before reaching the walls. Eventually, they spread throughout the building.



(d) Below are the pressure signals in the requested measurement points.



The thick solid lines show the envelope of the signal defined by its minima and maxima. The maximum achieved absolute values of the pressure are

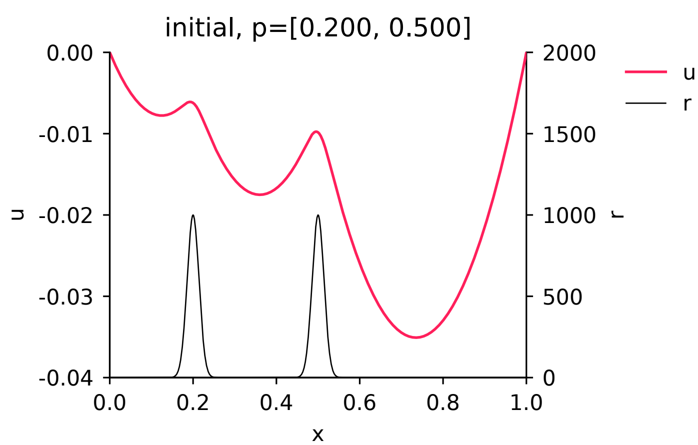
$$\max_n |p_A^n| = 2.45346831,$$
$$\max_n |p_B^n| = 0.20774921,$$
$$\max_n |p_C^n| = 0.15558662.$$

The values are consistent with the locations of the points, e.g. the pressure achieves higher values at point A located closer to the speaker. Note the effect of resonance: the pressure achieves values above 2.4 while the speaker imposes values only up to 1.

P2. Bridge designer

See solution code in `[p2_bridge.py]`.

(a) Below is the plot of the bridge height $u(x;p)$ and the coefficient $r(x,p)$ for $p = p_{\text{init}}$.



(b) The finite differences result in

$$\frac{\partial G}{\partial p_1} \approx -0.0194705086383612$$
$$\frac{\partial G}{\partial p_2} \approx -0.0351387025962491$$

(c) The direct method results in

$$\frac{\partial G}{\partial p_1} \approx -0.0194717549076273$$
$$\frac{\partial G}{\partial p_2} \approx -0.0351407098993236$$

(d) The adjoint method results in

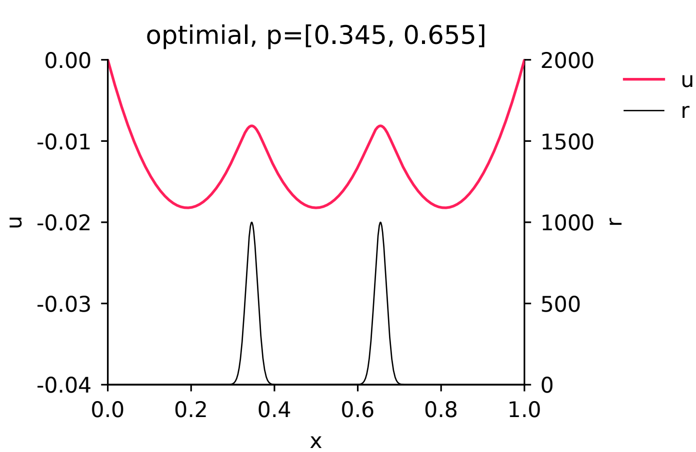
$$\frac{\partial G}{\partial p_1} \approx -0.0194717549076273$$
$$\frac{\partial G}{\partial p_2} \approx -0.0351407098993237$$

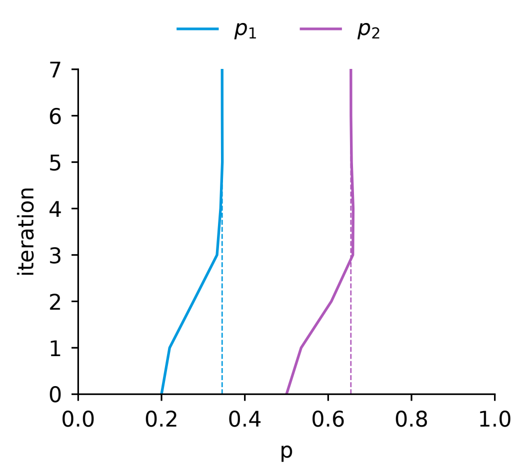
Note that the direct and adjoint methods produce values identical up to machine precision.

(e) Optimization using BFGS with derivatives from the adjoint method results in

$$p_1 = 0.3454631648894737$$
$$p_2 = 0.6545460569395484$$

The corresponding bridge height and the convergence history are shown below.





The pillars move to positions symmetric with respect to the center, and the average height of the bridge increases.

P3. Conjugate gradients

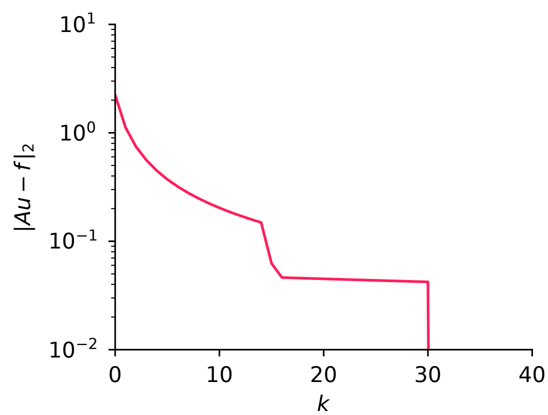
See solution code in [p3_cg.py].

(a) The first and last rows of $Au = f$ correspond to the boundary conditions $u_0 = a$ and $u_n = b$. The second and second last rows correspond to the equations with eliminated boundary conditions

$$-a + 2u_1 - u_2 = 0,$$
$$-u_{n-2} + 2u_{n-1} - b = 0.$$

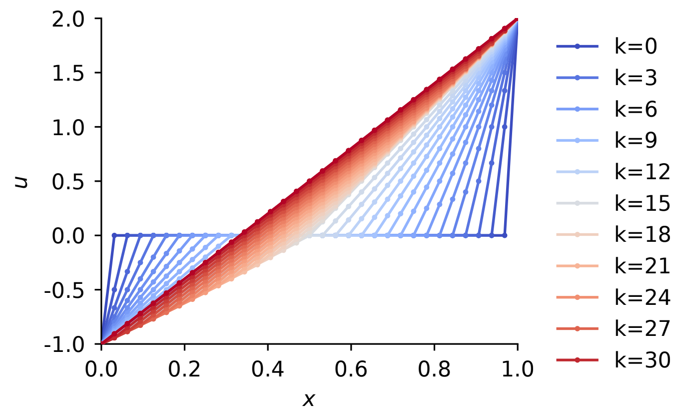
The other rows are equivalent to the equation away from the boundaries.

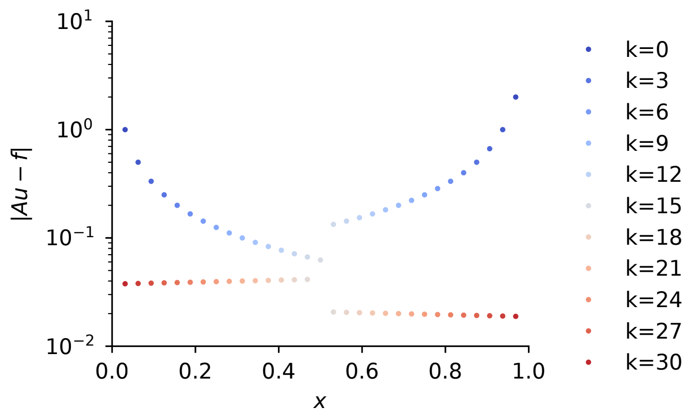
(b,c,d) Below is the convergence history of the method.



The residual behaves differently for $n < 15$ and $n > 15$, which is also consistent with part **(e)**. The residual quickly drops at about $n = 15$. The method takes 30 iterations to reduce the residual below 10^{-10}.

(e) The convergence history of the solution u and he corresponding residual are shown below. For every k, the residual takes values larger than 10^{-2} at only two points. Those two points propagate from the boundaries towards the center of the interval for $k < 15$ and then back for $k > 15$. This illustrates the local nature of the method: each iteration only involves multiplications by the matrix A, which contains at most three non-zero elements per row.


2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
u
0.0
0.2
0.4
0.6
0.8
1.0
x
k=0
k=3
k=6
k=9
k=12
k=15
k=18
k=21
k=24
k=27
k=30


10^{1}
10^{0}
10^{-1}
10^{-2}
$|Au - f|$
0.0
0.2
0.4
0.6
0.8
1.0
x
k=0
k=3
k=6
k=9
k=12
k=15
k=18
k=21
k=24
k=27
k=30