AM205 Quiz 1. Data fitting. Solution

Q1

What is the minimal degree of a polynomial passing through the following points

x	-1	0	1	2
y	1	0	1	4

☐ 1
☒ 2
☐ 3
☐ 4 or higher

Answer: Polynomial x^2 passes through the points.

Q2

How many polynomials of degree n pass through points $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ with distinct $x_1, \ldots, x_n$?

☐ zero
☐ one
☒ infinitely many
☐ depends on S

Answer: For any arbitrary value at one extra point, the polynomial of degree n would be unique.

Q3

Suppose that $f(x)$ is an interpolating spline of degree 5, i.e. $f(x)$ is a piecewise polynomial of degree 5 in each of n segments $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$ and $f(x)$ passes through points (x_i, y_i) for $i = 0, \ldots, n$. In addition, $f(x)$ is required to have 4 continuous derivatives in $[x_0, x_n]$. How many additional constraints need to be imposed on $f(x)$ to guarantee its uniqueness?

Answer: 4. There are $6n$ parameters, $2n$ interpolation constraints, $n - 1$ constraints on first derivative, $n - 1$ constraints on second derivative, $n - 1$ constraints on third derivative, and $n - 1$ constraints on fourth derivative.

Q4

Without accounting for the rounding error, which of the following polynomials interpolates discrete data more accurately?

☐ an interpolating polynomial constructed in the monomial basis?
☐ an interpolating polynomial constructed in the Lagrange polynomial basis?

☒ both have the same accuracy

Answer: The interpolation problem is solved exactly regardless of the basis chosen, and the two polynomials coincide.

Q5

Suppose that $L_i(x),\ i = 0, \ldots, n$ represent the Lagrange polynomial basis on points $\{x_i\}$. What is the degree of $g(x) = x\big(\sum_{i=0}^{n} L_i(x)\big)$?

Answer: 1. The sum $\sum_{i=0}^{n} L_i(x)$ is a polynomial of degree $\leq n$ that evaluates to 1 at $n+1$ points, so the sum is identical to 1, and $g(x) = x$ has degree 1.

Q6

Which of the following are examples of nonlinear least-squares fitting problems?

☐ fitting a straight line to discrete data
☐ fitting a quadratic polynomial *to* discrete data

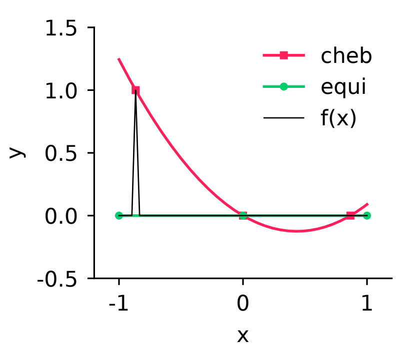
Answer: Both models depend linearly on the parameters, so the fitting problem is linear in both cases.

Q7

Interpolating polynomials are often used for function approximation. Two common choices for the set of interpolation points are the equidistant points and Chebyshev points. Which of the two will produce a lower approximation error?

☐ equidistant points
☐ Chebyshev points
☒ depends on the function

Answer: One would expect a lower error using Chebyshev points. However, this does not hold in general. Consider $f(x) = 0$ perturbed near one Chebyshev point.



Q8

Linear least-squares fitting is equivariant to scaling if fitting to data points (x_i, y_i) and $(\lambda x_i, y_i)$ results in functions $f(x)$ and $\tilde{f}(x)$ which are related as $f(x) = \tilde{f}(x/\lambda)$ for any $\lambda > 0$. Fitting in which of the following basis functions will be equivariant to scaling?

- ☒ monomials: 1, x, x^2
- ☒ Lagrange polynomials: $(x-2)(x-3)$, $(x-1)(x-3)$, $(x-1)(x-2)$
- ☐ exponentials: e^{-x}, 1, e^x
- ☒ logarithms: 1, $\log x$, $(\log x)^2$

Answer: Monomials and Lagrange polynomials both span all polynomials of degree ≤ 2. Scaling x in a polynomial yields another polynomial. Exponentials change in a non-linear way, see example in Unit 1. Under scaling λx, powers of the logarithm behave like polynomials under translation $x + \lambda$.

Q9

Suppose that the first derivative of $f(x) = \sin(x)$ is calculated at $x = 1$ using a finite difference approximation $\frac{f(x+h)-f(x)}{h}$ for an arbitrarily small $h > 0$. What is the best relative error you expect to achieve from a calculation with double precision?

- ☐ 1
- ☒ 10^{-8}
- ☐ 10^{-16}

Answer: The error will behave like $h + \epsilon/h$ which is minimized at $h = \sqrt{\epsilon}$. See Unit 0.

Q10

How many bits are used to represent a number in the IEEE double precision arithmetic?

Answer: 64

Q11

In the context of function approximation using a polynomial interpolant, what are the advantages of Chebyshev points over equidistant points?

- ☐ the degree of the polynomial grows slower with the number of points
- ☒ smaller error for the same number of points
- ☐ lower algorithmic complexity of evaluating the Lagrange polynomial

Answer: The error is generally lower, but the degree and the algorithmic complexity are the same.

Q12

Runge's function is an example of

- ☒ a smooth function for which the polynomial interpolation using equidistant points leads to an exponential growth of the infinity norm error
- ☐ a basis function equivariant to translation
- ☐ a continuous function with a discontinuous first derivative

Answer: Translation changes the function in a non-linear way, not compatible with equivariance to translation. Runge's function is smooth.

Q13

What is the maximum absolute value of a Chebyshev polynomial $T_{n+1}(x)$ in the range $[-1, 1]$?

- ☒ 1
- ☐ 2^{n+1}
- ☐ $1/2^{n+1}$

☐ $n+1$

Answer: From the definition $T_{n+1}(x) = \cos\left((n+1)\arccos x\right)$, the maximum value is 1.