AM205 Quiz 2. Numerical linear algebra. Solution

Q1

Which of the vector norm axioms are violated for the p-norm if $0 < p < 1$?

☐ absolute homogeneity
☒ triangle inequality
☐ positive definiteness
☐ none of the above

Q2

The product of two upper triangular matrices is an upper triangular matrix.

☒ true
☐ false

Q3

Consider a matrix $A \in \mathbb{R}^{n \times n}$ and vector $b \in \mathbb{R}^n$. Assume that an LU factorization of A is known. What is the complexity of solving the linear system $Ax = b$ using that LU factorization?

☐ $\mathcal{O}(n)$
☒ $\mathcal{O}(n^2)$
☐ $\mathcal{O}(n^3)$
☐ none of the above

Q4

Let L_j be an elementary elimination matrix from one step of the LU factorization algorithm for a square matrix A. Which of the following statements are correct in general for any A? The matrix L_j is

☒ invertible
☒ lower triangular
☐ orthogonal
☒ sparse
☐ none of the above

Answer: The matrix is not orthogonal since its inverse is obtained by negating the elements below the diagonal, which is different from its transpose.

Q5

Suppose that a square matrix A has a Cholesky factorization $A = LL^T$, where L is a square invertible lower triangular matrix. Which of the following statements are correct in general for any L? The matrix A is

☐ lower triangular
☒ positive-definite
☒ symmetric
☐ none of the above

Q6

Which of the following factorizations of a square matrix are unique?

☐ LU
☐ QR
☒ none of the above

Q7

Suppose that F is a Householder reflector. Which of the following statements are correct in general?

☒ F is orthogonal
☒ $F^2 = I$
☐ none of the above

Q8

Suppose that Q is an orthogonal matrix and $Q = U\Sigma V^T$ is its singular value decomposition. Which of the following statements are correct in general?

☒ Σ is diagonal
☒ Σ is invertible
☒ $\|\Sigma\|_2 = 1$
☐ none of the above

Answer: For any Q, $\|\Sigma\|_2 = \|Q\|_2$. Since Q is orthogonal, $\|Q\|_2 = 1$.

Q9

Consider a matrix $A \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$. Which of the following factorizations, once known, reduce the complexity of solving the linear system $Ax = b$ to $\mathcal{O}(n^2)$?

☒ LU
☒ QR
☒ SVD
☒ Cholesky
☐ none of the above

Answer:

- $A = LU$, solve $Ly = b$, $Ux = y$
- $A = QR$, solve $Qy = b$, $y = Q^T b$, $Rx = y$

- $A = U\Sigma V^T$, solve $Uy = b$, $y = U^T b$, $\Sigma V^T x = y$, $V^T x = \Sigma^{-1} y$, $x = V\Sigma^{-1} y$
- $A = LL^T$ is a type of LU