AM205 Quiz 4. Optimization. Solution

Q1

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear smooth function with a fixed point $\alpha \in \mathbb{R}$, i.e. $g(\alpha) = \alpha$. Which of the following statements are true in general?

- ☐ $g'(\alpha) = 1$
- ☐ $|g'(\alpha)| < 1$
- ☐ $|g'(\alpha)| \le 1$
- ☒ none of the above

Answer: For example, consider a function $\hat{g}(x) = g(x) + c(x - \alpha)$ which has the same fixed point, but can have an arbitrary derivative depending on c.

Q2

Suppose that a sequence x_k converges linearly to α. Define $y_k = (x_k - \alpha)^2$. Which of the following statements is true in general?

- ☒ y_k converges linearly to 0
- ☐ y_k converges superlinearly to 0

Answer: Linear convergence means $\lim_{k\to\infty} |x_{k+1} - \alpha|/|x_k - \alpha| = \mu$, where $0 < \mu < 1$. For y_k, we have $\lim_{k\to\infty} |y_{k+1}|/|y_k| = \lim_{k\to\infty} (x_{k+1} - \alpha)^2/(x_k - \alpha)^2 = \mu^2$, so y_k converges linearly to 0.

Q3

Consider a scalar equation $f(x) = 0$ with a smooth and strictly convex function $f : \mathbb{R} \to \mathbb{R}$. Which of the following methods are expected to converge **superlinearly**? Assume that the initial guess is chosen sufficiently close to a solution.

- ☐ bisection method
- ☒ Newton's method
- ☒ secant method
- ☐ none of the above

Q4

Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$. Which of the following statements are true?

- ☒ if f is coercive on $\mathbb{R}$, then f has a global minimum in $\mathbb{R}$
- ☐ if f has a unique global minimum in $\mathbb{R}$, then f is coercive on $\mathbb{R}$
- ☐ none of the above

Q5

The function $f(x) = |x|$ defined on $\mathbb{R}$ is

☒ coercive
☒ convex
☐ strictly convex
☐ none of the above

Q6

The Hessian of the function $f(x, y) = x^2 + y^2$ is

☒ positive definite
☐ negative definite
☐ indefinite
☐ none of the above

Q7

To optimize a function $f : \mathbb{R}^n \to \mathbb{R}$, the BFGS algorithm relies on evaluations of

☐ the function f
☒ the gradient ∇f
☐ the Hessian H_f

Q8

Recall the Lagrangian function $\mathcal{L}(b, \lambda) = b^T b + \lambda^T (Ab - y)$ corresponding to an underdetermined linear least squares problem. Assume that $A \in \mathbb{R}^{m \times n}$ has full rank and $m \leq n$. Suppose that this function is minimized using Newton's method with a zero initial guess $b_0 = 0$ and $\lambda_0 = 0$. How many iterations would Newton's method need to satisfy $\|\nabla \mathcal{L}\|_2 < 10^{-5}$?

☒ one
☐ depends on $\|A\|_2$
☐ depends on $\|A\|_2$ and $\|y\|_2$

Answer: Newton's method solves any quadratic optimization problem exactly after one iteration.